Transient 1/*f* noise

Sudeshna Sinha
Indian Institute of Astrophysics, Koramangala, Bangalore 560 034, India
(Received 10 July 1995)

We provide examples of transient (nonasymptotic) ''1/*f* noise,'' occurring over reasonably long time scales, in extended nonlinear systems modeling complex spatiotemporal phenomena. These studies contribute towards establishing that transient phenomena, which is often quite relevant in numerical and laboratory experiments, can hold a wealth of interesting dynamical features.

PACS number(s): 05.45.+b

I. INTRODUCTION

The asymptotic features of dynamical systems have long been emphasized and the use of long time averages is very common in the definition of various dynamical concepts, such as periodicity, intermittency, and chaos. Here we would like to focus on the very interesting, though much ignored, physics of *transient states*. Transient signals are traditionally discarded, as they are considered merely switching on or warming up effects. In numerical experiments, transients, even those that are rather long, are not analyzed on the same footing as the stable asymptotics. Usually one merely notes the presence of transience, long or short, without analyzing the *dynamical characteristics of the transience.*

However, transience, which can also be viewed as a ''nonequilibrium'' phenomenon, can actually have rather exciting dynamical features, closely related to many aspects of nonlinear dynamics. It is characterized by the system moving around, for instance, ''chaotically'' or ''intermittently,'' and then suddenly settling down to a stable dynamical behavior. Moreover, recent work on transient chaos in a NMR laser shows that the physics of transience is accessible experimentally [1], and this motivates us further to research transience with a new outlook.

In this paper we have investigated the dynamical features of transience arising in numerical simulations of certain model spatiotemporal systems. We have brought to light some transient dynamics (rather long lived) displaying 1/*f* spectral characteristics up to frequencies that are rather small. We hope our studies contribute towards establishing transient dynamics as being rich in phenomena, and thus providing a fertile ground for future numerical and experimental investigations.

II. EXAMPLES OF TRANSIENT 1/*f* NOISE

Low-frequency noise has long attracted theoretical model builders as there is a lot of evidence to indicate that it is quite ubiquitous in nature, occurring widely as it does in a host of composite processes, as diverse as star-flicker in astronomy, flow of sand in an hour glass, to traffic movement and stock market fluctuations [2]. It is therefore of immense interest to construct paradigmatic models of dynamic processes that are capable of yielding 1/*f* spectra [3]. Here we suggest that there are situations where *transient phenomena have 1/f-like characteristics*. The 1/*f*-like features arise from transient ''intermittent'' behavior, which can also be viewed as an ''intermittent route'' to the asymptotic state. Note also that the ''1/*f*'' behavior can persist to rather low frequencies (but cannot approach $f=0$, as transience is necessarily finite). However, since in natural systems and in laboratory experiments one can only sample up to finite times, it is not always entirely clear if the dynamics is the asymptotic state or a reasonably long-lived transient. In this context the presence of transient 1/*f* noise is rather pertinent.

Here we provide two examples of transient 1/*f* noise, occurring in extended nonlinear systems relevant to complex spatiotemporal phenomena [4]. Finite time series Fourier analysis of the (long-lived) transience yields a many decade 1/*f*-like spectral density. We discuss the details below.

A. Adaptive lattice dynamics

The first example is obtained from adaptive lattice dynamics. This is a class of models proposed recently [5,6] that incorporates an adaptive dynamics on a lattice of strongly nonlinear elements. In these models time is discrete, labeled by n, space is discrete, labeled by $i, i=1,\ldots,N$ where N is system size, and the state variable $x_n(i)$ (which in physical systems could be quantities such as energy or pressure) is continuous. Each individual site in the lattice evolves under a suitable nonlinear map $f(x)$. Here we choose $f(x)$ to be the logistic map, which has widespread relevance as a prototype of chaos. So $f(x)=1-ax^2, x=[-1,1]$, with the nonlinearity parameter a chosen in the chaotic regime ($a=2.0$ in all subsequent numerics). On such a lattice a regulatory threshold mechanism is incorporated. This is triggered when a site in the lattice exceeds the critical value x_c, i.e., when a certain site $x_n(i)>x_c$ [7]. The supercritical site then relaxes (or ''topples'') by transporting its excess to its neighbors. For the bidirectional model this occurs as follows:

$$x_n(i)\rightarrow x_c,$$
$$x_n(i+1)\rightarrow x_n(i+1)+\delta x,$$
$$x_n(i-1)\rightarrow x_n(i-1)+\delta x, \tag{1}$$

where $\delta x=[x_n(i)-x_c]/2$. This algorithm thus induces nonlinear transport along the array (by initiating a domino effect). The boundary is open so that the ''excess'' may be transported out of the system. This kind of threshold mechanism imposed on local chaos makes the above scenario es-

 © 1996 The American Physical Society

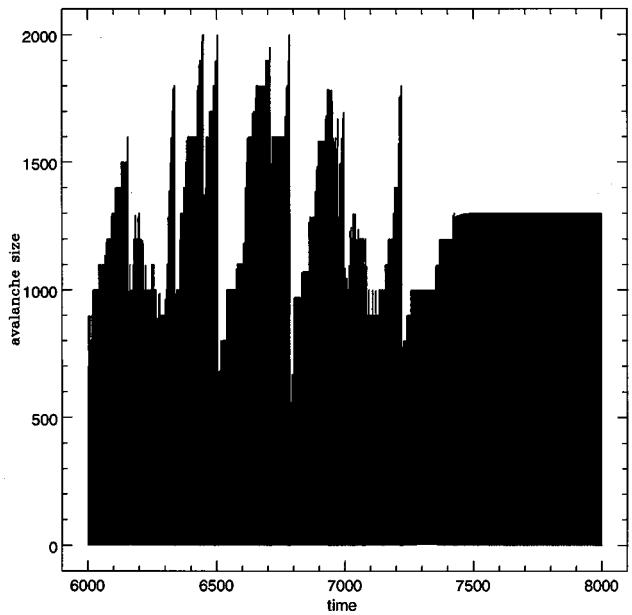

FIG. 1. Plot of avalanche size vs time n ($n=6000$–8000), for the case of $x_c=0.98$, in a bidirectional adaptive model on a lattice of logistic maps. Note the abrupt transition to the asymptotic regular state (at $n\sim 7700$).

pecially relevant for certain mechanical systems such as chains of nonlinear springs, as well as for some biological systems, such as synapses of nerve tissue (note that individual neurons display complex chaotic behavior and have step-function-like responses to stimuli).

Dynamical quantities of interest include local quantities such as the on-site state variable $x_n(i)$ for specific i, as well as global quantities such as the size of ''avalanches,'' defined as the total number of ''active'' sites, i.e., sites that have ''moved'' (or dissipated energy) during the adaptive relaxation [5,6]. Here we will be concentrating, with no loss of generality [8], on the dynamical evolution of avalanche sizes.

The most significant parameter in the model is the critical x_c and the dynamics is principally determined by it [5,6]. Tuning x_c leads to the emergence of a variety of ''phases'' in x_c space, characterized by a rich diversity of dynamical patterns, starting from fixed points to windows of regular or noisy cycles of all orders.

Here we focus on values of critical x_c lying in reasonably wide windows of periodicity, where the asymptotic dynamics goes to stable regular cycles. It was found that when the parameters were close to the edge of the window, the transience of these periodic states was very long lived ($\sim 10^4$). Further the transient dynamics appeared intermittent (see Fig. 1), and spectral analysis of this (finite) time series showed distinct $1/f^{\phi}$ scaling at the low-frequency end, with $1<\phi<1.5$. We illustrate this with two representative examples in Fig. 2: (a) $x_c=0.89$ (in the five-cycle window), and (b) $x_c=0.98$ (in the four-cycle window). Note that a large peak occurs at the frequency of the asymptotic stable cycle, in the spectra of the transient dynamics, in addition to the $1/f$-like features of the low-frequency end.

It was found that the transience was easily distinguishable from the asymptotics, and the change from transient behavior

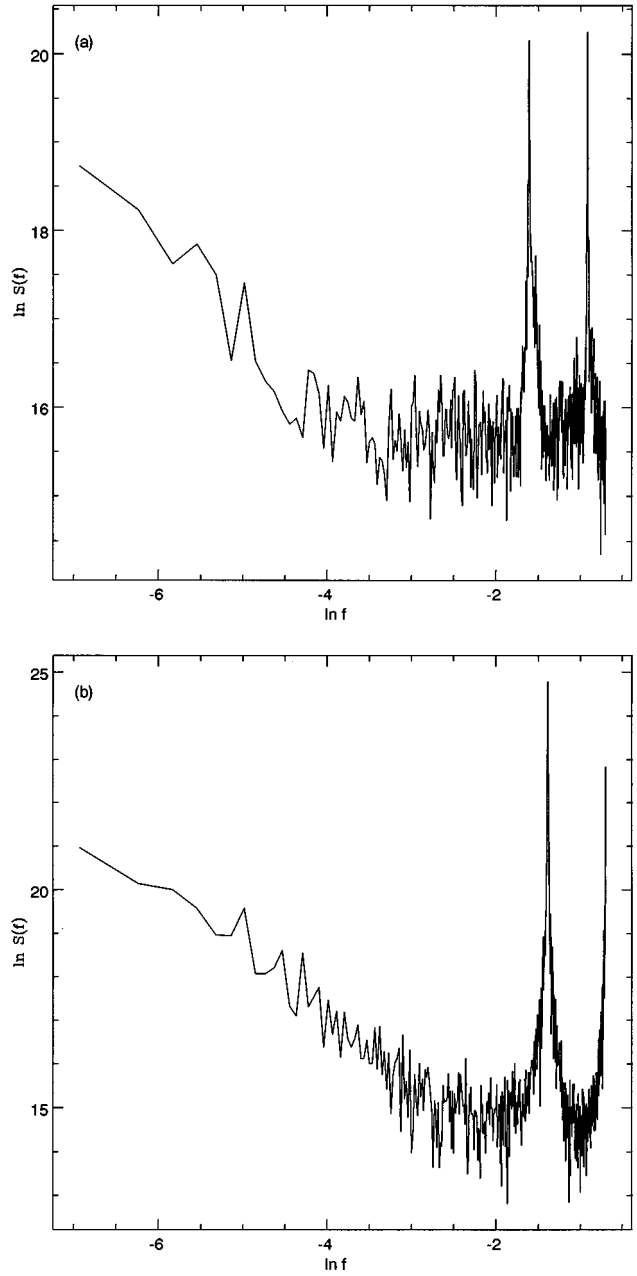

FIG. 2. Power spectrum of avalanches in a bidirectional adaptive model on a lattice of logistic maps, for critical $x_c=$(a) 0.89 and (b) 0.98 ($N=20$ in both cases). Here we average over 8 transient pieces of 1024 each. The x axis has $\ln f$, where f is the frequency ($f\in(0,0.5]$), and the y axis has $\ln S(f)$, where $S(f)$ is the power.

to stable behavior was sudden and abrupt (see Fig. 1). Also note that the smoothing procedure employed to obtain suitably ''averaged'' spectra involve *ensemble averages*. That is, several relaxations of the dynamics (evolved from different sets of initial conditions) are collected and the transient pieces (each of which is reasonably long) are independently Fourier transformed and the spectra thus obtained are averaged. We do not, as in Ref. [1], construct a long *artificial* time series by gluing smaller stretches of signal, and then analyzing it. In the process we eliminate noise errors due to the gluing procedure, and also obtain a very natural easily interpretable physics from the spectra. Of course the smallest

frequency resolution of the spectra is limited by the length of the transient, and so the analysis does not make much sense unless the transience is reasonably long. (In the examples quoted here this does not pose a problem as the transient lengths are very large.) Now in the section below, we will discuss another example of transient $1/f$ noise, occurring now in a hierarchical lattice.

B. Hierarchically coupled maps

Turbulent convective transport is characterized by the coupling between fluctuations on different length scales. There is a chaotic cascade of velocity fluctuation, which terminates finally on a length scale where diffusion of momentum brought about by the viscosity of the fluid eliminates velocity gradients sufficiently high for further instability. This self-organization of chaotic fluctuations leads to an essentially conservative transport of kinetic energy from large to small scales known as the Richardson-Kolmogorov cascade.

A system of coupled maps was proposed recently by Sinha and Thomae [9] to mimic some of these features found in fully developed turbulence. In contrast to the widely discussed coupled-map lattices, this lattice of logistic maps was constructed with *hierarchical* coupling to account for the peculiarities of turbulent fluid flow.

The elementary building block of the model is again the logistic map (written now in a form different from that in Sec. II A, but equivalent by simple scaling):

$$p_{\tau+1}=ap_\tau(1-p_\tau),\quad 0\leqslant p\leqslant 1,\quad 0\leqslant a\leqslant 4. \tag{2}$$

Here $\tau=0,1,2,\ldots$ denotes the discrete time, p is the dynamical variable, and a the control parameter. Equation (2) provides the elementary building block for the construction of our model, which defines the dynamics of variables $P(\mathbf{r},\tau)$, describing fluctuations in space ($\mathbf{r}$) and time (τ). $\mathbf{r}\in\mathbb{Z}^d$ denotes points on a d-dimensional cubic lattice.

At each site P is decomposed into fluctuating contributions $p_\tau^{(n)}(\mathbf{r})$ from hierarchical levels n, $0\leqslant n\leqslant N$. The fluctuations on level n, $n=0,1,2,\ldots,N$, have a typical length scale $l^{(n)}$ and a typical time scale $\vartheta^{(n)}$. We associate with the largest n the smallest scales. For simplicity we choose

$$l^{(n)}=2l^{(n+1)},\quad \vartheta^{(n)}=2\vartheta^{(n+1)}. \tag{3}$$

This means that the same value of $p^{(n)}$ applies to a cubic cell of $(2^{N-n})^d$ sites and is updated only once when $p^{(n+1)}$ is updated twice.

The time evolution of the $p^{(n)}$'s is defined essentially by coupling the control parameter of the logistic map to the hierarchical and spatial distribution of P. We assume that the dynamics depends on the spatial distribution of $P(\mathbf{r},\tau)$ only via gradients or higher derivatives with respect to $\mathbf{r}$. Thus, the dynamics of spatially homogeneous solutions will depend only on the coupling between different levels n and from now on we will concentrate only on this case.

With the idea that the dynamical variable p_τ in Eq. (2) represents something like a local magnitude of the velocity gradient in a flow field we introduce the hierarchical coupling by

$$a_\tau^{(n)}:=4-(1/q)(1-p_{\tau+1}^{(n-1)}),\quad 1/4\leqslant q<\infty, \tag{4}$$

i.e., we assume that the control parameter at level n is a linear function of the dynamical variable at level $n-1$. The higher the ''velocity gradient'' on level $n-1$ the more unstable and chaotic is the dynamics on level n. The coupling parameter $1/q$ represents something like a local critical Reynolds number. The larger $1/q$, the higher the value of $p^{(n-1)}$ necessary to obtain a certain degree of instability on level n.

Thus the dynamical equation on level n reads

$$p_{\tau+1}^{(n)}=\begin{cases}\{4-(1/q)(1-p_{\tau+1}^{(n-1)})\}p_\tau^{(n)}(1-p_\tau^{(n)}) & \text{if } \tau\equiv 0(\text{mod } 2^{(N-n)})\\ p_\tau^{(n)} & \text{otherwise .}\end{cases} \tag{5}$$

Note the dependence of a_τ on $p_{\tau+1}$ in Eq. (4). In the context of Eq. (5) this means that each time there is a change on level n_0 all levels n with $n>n_0$ will be updated as well. In a system with N levels, then, updates on level n are due at

$$\tau^{(n)}(\sigma)=2^{N-n}\sigma,\quad \sigma=0,1,2,\ldots. \tag{6}$$

We call σ the reduced time on level n.

Spatially, our model consists of a nested hierarchy of d-dimensional cubes. Level-n cubes contain $(2^{N-n})^d$ sites. Each level-n cube contains 2^d level-$(n+1)$ cubes.

We now briefly give some of the interesting analytical properties we can derive for our model [9].

The static solution $p_*^{(n)}$ of Eq. (5) has to satisfy the condition $p_{\tau+1}^{(n)}=p_\tau^{(n)}$ for all τ and n as well as the boundary condition for $n=0$. Hence Eq. (5) implies the recursion relation

$$p_*^{(n)}=1-\frac{1}{4-(1/q)(1-p_*^{(n-1)})}. \tag{7}$$

In the limit $n\to\infty$ the sequence $p_*^{(n)}$ converges to the alternating continued fraction

$$p_*^{(\infty)}=1+\cfrac{1}{-4+\cfrac{1}{4q+\cfrac{1}{-4+\cdots}}}=[1,-4,4q,-4,4q,\ldots]$$

This yields the value of $p_*^{(\infty)}$ to be

$$p_*^{(\infty)}=(1-2q)+\sqrt{(4q-1)q}. \tag{8}$$

Clearly, the lower bound for q is 1/4.

Further one obtains the corresponding relations for the control parameter

$$a_*^{(n)}=4-\frac{1}{qa_*^{(n-1)}}, \quad a_*^{(\infty)}=2+\sqrt{4-1/q}. \tag{9}$$

Since the control parameter at each level must be within [0,4] the recursion requires that $1/4q \leqslant a_* \leqslant 4$. For $a_*^{(\infty)}$ one has $2 \leqslant a_*^{(\infty)} \leqslant 4$.

It can also be shown rigorously that the rate of convergence of $p_*^{(n)}$ to $p_*^{(\infty)}$ is characterized by a scaling relation:

$$\delta := \lim_{n\to\infty} \frac{p_*^{(n+1)}-p_*^{(n)}}{p_*^{(n+2)}-p_*^{(n+1)}} = 8q-1+4\sqrt{(4q-1)q}. \tag{10}$$

Further, linear stability of the static solution gives the eigenvalues of the system to be

$$\lambda_*^{(n)}=-2+(1/q)(1-p_*^{(n-1)}), \quad n=1,2,\ldots,N.$$

Since $0 \leqslant p_* \leqslant 3/4$ the range for all eigenvalues is $-2+1/4q \leqslant \lambda_*^{(n)} \leqslant -2+1/q$. $|\lambda_*^{(n)}|<1$ indicates stability and $|\lambda_*^{(n)}|>1$ linear instability of the nth eigenmode. Negative eigenvalues mean that the corresponding mode oscillates while positive values indicate monotonic behavior.

This implies the following recursion relation and the stable fixed point for the eigenvalues:

$$\lambda_*^{(n+1)}=-2+\frac{1}{q(2-\lambda_*^{(n)})}, \quad \lambda_*^{(\infty)}=-\sqrt{4-1/q}. \tag{11}$$

Evidently, $\lambda_*^{(\infty)}<0$ if $1/4<q$. Hence, for all n sufficiently large, the eigenmodes will show oscillatory behavior.

Here we will consider the limiting case of $q=1/4, a_*^{(\infty)}=2$, where the evolution equation is $p_{\tau+1}^{(n)}=4p_\tau^{(n-1)}p_\tau^{(n)}(1-p_\tau^{(n)})$, with boundary condition $p_\tau^{(0)}=1$. This yields the following scenario: levels $n>1$ [10] evolve to the stable fixed point, $p^{(n)}=0$ [11]. But for n close to 1, this asymptotic state is reached only after a very long transience (the transience is shorter for finer scales, i.e., larger n). For example, then, we find that $p^{(2)}$ has a transience of length $\sim 10^3$ or 10^4, after which it abruptly settles down to the asymptotic fixed point $p^{(2)}=0$. Further the transient dynamics is strongly reminiscent of intermittency, as it displays long laminar periods, where $p^{(2)}$ is close to 0, interrupted ''intermittently'' by bursts of irregularity (see Fig. 3). This gives rise to a power spectrum with the low-frequency end scaling as $1/f^\phi$, with $\phi \sim 1$ (see Fig. 4). Note that here too, as described in Sec. II A, ensemble averages were used to obtain suitably averaged spectra.

In conclusion, we have furnished examples of intermittent transient dynamics giving rise to spectra with $1/f$-like features up to rather low frequencies, in extended nonlinear systems. We hope these studies contribute towards establishing that transient phenomena can hold a wealth of interesting dynamical features.

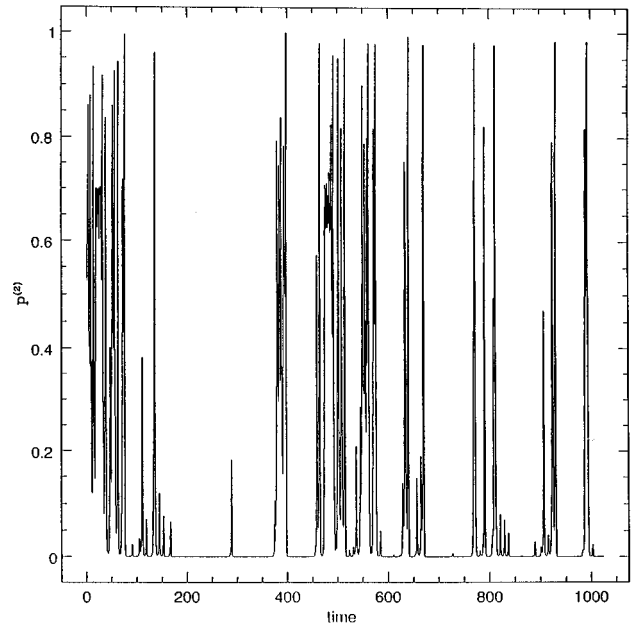

FIG. 3. Time evolution of $p^{(2)}$ in hierarchically coupled maps, for the case of $q=1/4, a^{(\infty)}=2.0$ during transience (i.e., before the system reaches its asymptotic state: $p^{(2)}=0$). Note the intermittent behavior.

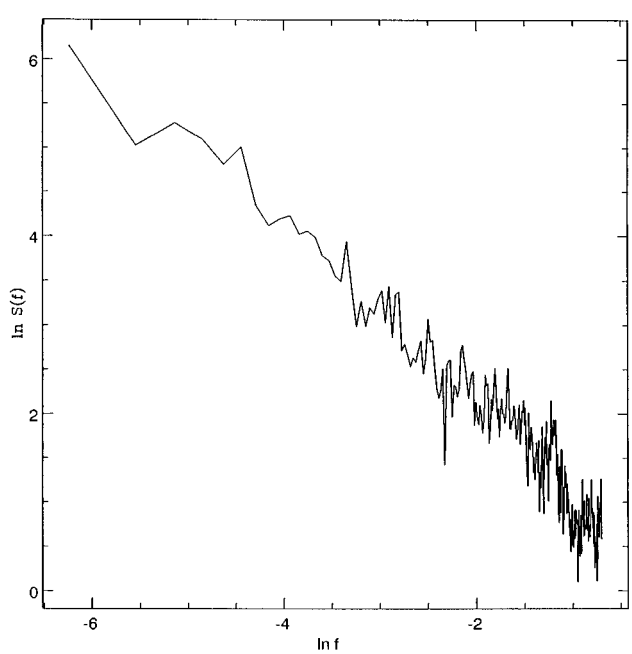

FIG. 4. Power spectrum of $p^{(2)}$ in hierarchically coupled maps, for $q=1/4$, $a^{(\infty)}=2.0$. We average over 8 transient pieces of 1024 each. The x axis has $\ln f$, where f is the frequency ($f \in (0,0.5]$), and the y axis has $\ln S(f)$, where $S(f)$ is the power.

[1] I. Janosi, L. Flepp, and T. Tel, Phys. Rev. Lett. **73**, 529 (1994).
[2] W. H. Press, Comments Mod. Phys. C **7**, 103 (1978); P. Dutta and P. M. Horn, Rev. Mod. Phys. **53**, 497 (1981); M. B. Weissman, *ibid.* **60**, 537 (1988); *Noise in Physical Systems*, Proceedings of the Fifth International Conference on Noise, edited by D. Wolf (Springer-Verlag, Berlin, 1978).
[3] The noise is called $1/f$ despite the fact that the exponent is rarely 1, but varies from system to system and is typically in the range 0.6–1.6.
[4] J. Crutchfield and K. Kaneko, in *Directions in Chaos*, edited by Hao Bai-Lin (World Scientific, Singapore, 1987), and references therein.
[5] S. Sinhg and D. Biswas, Phys. Rev. Lett. **71**, 2010 (1993).
[6] S. Sinha, Phys. Rev. E **49**, 4832 (1994); Phys. Lett. A **199**, 365 (1995); Int. J. Mod. Phys. B **9**, 875 (1995).
[7] Note that there is no external noise in the model.
[8] The spectral characteristics of the local site dynamics is very similar to that of various global quantities.
[9] S. Sinha and S. Thomae (unpublished).
[10] The grossest level $n=1$ has a grassy power spectrum, which is expected since $p^{(0)}=1$ and so level $n=1$ reduces to the fully chaotic logistic map at $a=4$.
[11] When $q\sim 1/4$ the nonlinearity parameter at level $n+1\sim 4p^{(n)}$. So it is easy to see that once a certain level n has been attracted to the fixed point at $p^{(n)}=0$ all the levels below it $n+1, n+2, \ldots, N$ will go to the field point at 0.